finishing § 3.4

similarity ($A \sim B$) is an equivalence relation:

- reflexive: $n \times n$ matrix A is similar to itself $A \sim A$
- symmetric: $A \sim B \rightarrow B \sim A$
- transitive: $A \sim B$ and $B \sim C \rightarrow A \sim C$

also, $A \sim B \rightarrow A^2 \sim B^2$

$A^3 \sim B^3$

$\vdots$

$A^n \sim B^n$

Diagonalization

given A is $n \times n$ matrix let λ_i be scalar

lambda

$T(\vec{x}) = A\vec{x}$ $\mathbb{R}^n \rightarrow \mathbb{R}^n$

$\mathcal{B} = (\vec{v}_1, \ldots \vec{v}_n)$

basis of $\mathbb{R}^n$

if $T(\vec{v}_1) = \lambda_1 \vec{v}_1 = \begin{bmatrix} \vec{v}_1, & \ldots, & \vec{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

if $T(\vec{v}_2) = \lambda_2 \vec{v}_2 = \begin{bmatrix} \vec{v}_1, & \ldots, & \vec{v}_n \end{bmatrix} \begin{bmatrix} 0 \\ \lambda_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\vdots$

if $T(\vec{v}_n) = \lambda_n \vec{v}_n = \begin{bmatrix} \vec{v}_1, & \ldots & \vec{v}_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \lambda_n \end{bmatrix}$

then $B = S^{-1}AS$ is a diagonal matrix iff $T(\vec{v}_1) = \lambda_1 \vec{v}$

$T(\vec{v}_2) = \lambda_2 \vec{v}_2$

$\vdots$

$T(\vec{v}_n) = \lambda_n \vec{v}_n$

$\mathcal{B}$-matrix B of T

$$B = \begin{array}{c c} \begin{matrix} T(\vec{v}_1) & T(\vec{v}_2) & & T(\vec{v}_n) \end{matrix} & \\ \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} & \begin{matrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{matrix} \end{array}$$

$$D = \begin{bmatrix} \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{matrix} \vdots \\ \vec{v}_n \end{matrix}$$

can phrase as : A is <u>diagonalizable</u> when $\exists\ \mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$

$$\text{s.t. } S^{-1}AS = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \leftarrow \text{is a diagonal matrix}$$

not covering Ch. 4

§ 5.1

seen linear transformations : • orthogonal projection
§2.2 • reflections
• rotations

next, generalize orthogonal projection onto subspace V on $\mathbb{R}^n$

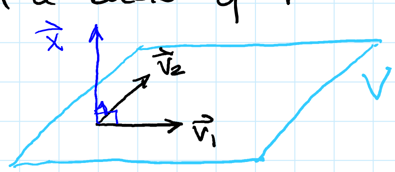
recall:

- $\vec{v}, \vec{w} \in \mathbb{R}^n$ are orthogonal when $\vec{v} \cdot \vec{w} = 0$
- length or norm of $\vec{v} \in \mathbb{R}^n$ is: $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$
- $\vec{u} \in \mathbb{R}^n$ is unit vector : $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = 1$
- if $\vec{v}$ is non-zero then $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = 1$

$\vec{x} \in \mathbb{R}^n$ is orthogonal to subspace V of $\mathbb{R}^n$
if $\vec{x}$ is orthogonal to all $\vec{v}$ in subspace V
i.e. $\vec{x} \cdot \vec{v} = 0$

if given basis $\vec{v}_1, \dots, \vec{v}_m$ of V then $\vec{x}$ is orthogonal to V
iff $\vec{x}$ is orthogonal to all vector $\vec{v}_1, \dots, \vec{v}_m$

ex. $\vec{x} \in \mathbb{R}^3$ is orthogonal to plane V in $\mathbb{R}^3$ iff $\vec{x}$ is orthogonal to $\vec{v}_1, \vec{v}_2$ that form a basis of V



$\vec{x}$ is orthogonal to both $\vec{v}_1$ and $\vec{v}_2$

def'n : $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m \in \mathbb{R}^n$ are called <u>orthonormal</u>

i.e., they're orthonormal if unit vectors are orthogonal to one another

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

ex. $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \in \mathbb{R}^n$ are orthonormal

ex. for scalar θ, show $\underbrace{\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}}_{\vec{v}_1}$ and $\underbrace{\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}}_{\vec{v}_2}$ are orthonormal

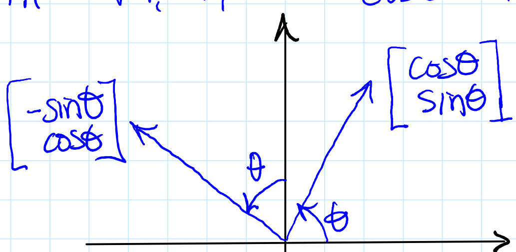
1. show dot product is 0:

$$\vec{v}_1 \cdot \vec{v}_2 = 0$$

$$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \cdot \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = (\cos\theta)(-\sin\theta) + \sin\theta\cos\theta = 0$$

2. show vectors have length of 1:

$$\|\vec{v}_1\| = \sqrt{\vec{v}_1 \cdot \vec{v}_1} = \cos\theta^2 + \sin^2\theta = 1 \qquad \text{likewise } \|\vec{v}_2\| = 1$$



ex. show $\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$ in $\mathbb{R}^4$ are orthonormal

1. $\vec{u}_1 \cdot \vec{u}_2 = 0$

$$\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) = 0$$

likewise, conclude $\vec{u} \cdot \vec{u}_3 = 0$
$\vec{u}_2 \cdot \vec{u}_3 = 0$

2. $\|\vec{u}_i\| = 1$ $\quad \|\vec{u}_1\| = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2} = 1$

likewise $\|\vec{u}_2\| = \|\vec{u}_3\| = 1$

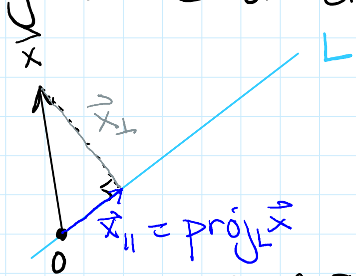
$\therefore \vec{u}_1, \vec{u}_2, \vec{u}_3$ are orthonormal in $\mathbb{R}^4$

<u>Properties</u>:
- orthonormal vectors are linearly independent
- " $\vec{u}_1, \ldots, \vec{u}_n \in \mathbb{R}^n$ form a basis of $\mathbb{R}^n$

recall orthogonal projection construction in $\mathbb{R}^2$

$\vec{x}$

recall orthogonal projection construction in $\mathbb{R}^2$



next, generalize to any subspace V of $\mathbb{R}^n$

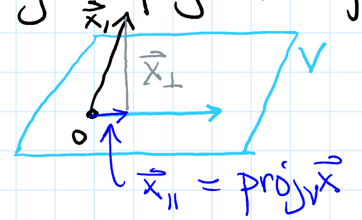
thm: given $\vec{x} \in \mathbb{R}^n$ and subspace V of $\mathbb{R}^n$ then can write $\vec{x}$ as:

$$\vec{x} = \vec{x}_\parallel + \vec{x}_\perp \quad \text{where } \vec{x}_\parallel \text{ is in } V, \; \vec{x}_\perp \text{ is orthogonal to } V$$

this representation is unique

also, $\vec{x}_\parallel$ is the orthogonal projection of $\vec{x}$ onto V: $\text{proj}_V \vec{x}$

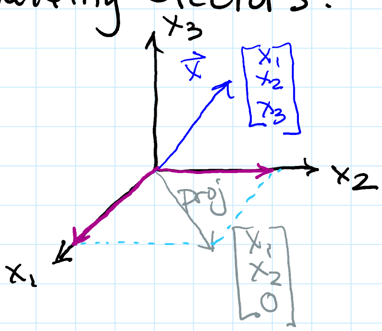
update construction



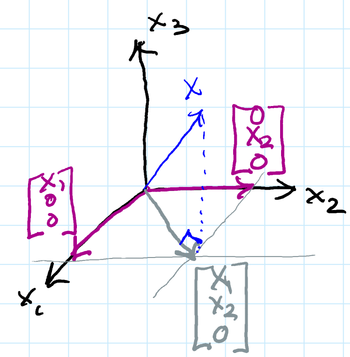
thm: if V is a subspace of $\mathbb{R}^n$ with an orthonormal basis $\vec{u}_1, \dots, \vec{u}_m$

then
$$\boxed{\text{proj}_V \vec{x} = \vec{x}_\parallel = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \dots + (\vec{u}_m \cdot \vec{x})\vec{u}_m \quad \forall \vec{x} \in \mathbb{R}^n \\ = \sum_{i=1}^{m} (\vec{u}_i \cdot \vec{x})\vec{u}_i}$$

ex. project a vector in $\mathbb{R}^3$ onto x_1-x_2 plane – would be same as projecting it to x_1-axis then onto x_2-axis then adding the resulting vectors:



vs.



ex. given subspace $V = \text{im}\, A$ of $\mathbb{R}^4$ where $A = \begin{array}{cc} \left[\begin{matrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{matrix}\right] \\ \vec{v}_1 \quad \vec{v}_2 \end{array}$, find $\text{proj}_V \vec{x}$ given $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 7 \end{bmatrix}$

sol'n: column vectors of A are linearly independent (since form an image of A)

$\therefore$ they form a basis of V

(since form an image of A)
$\therefore$ they form a basis of V

next, show $\vec{v}_1, \vec{v}_2$ form an <u>orthonormal</u> basis:

1 $\vec{v}_1 \cdot \vec{v}_2 = 0 \Rightarrow \vec{v}_1, \vec{v}_2$ are orthogonal

2. $\|\vec{v}_1\| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2$ not unit vector

get $\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$ likewise, $\vec{u}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$

$$\text{proj}_V \vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2$$

$$\underbrace{\left(\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 1 \\ 7 \end{bmatrix}\right)}\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + \underbrace{\left(\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 1 \\ 7 \end{bmatrix}\right)}\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$= \frac{1+3+1+7}{2}(\vec{u}_1) + \frac{1-3-1+7}{2}(\vec{u}_2)$$

$$= 6\vec{u}_1 + 2\vec{u}_2$$

$$= 6\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + 2\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 4 \end{bmatrix} = \text{proj}_V \vec{x}$$

check:

$\vec{x} = \text{proj}_V \vec{x} + \vec{x}_\perp$

$\vec{x} - \text{proj}_V \vec{x} = \vec{x}_\perp$

$$\begin{bmatrix} 1 \\ 3 \\ 1 \\ 7 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$\vec{x}_\perp \cdot \vec{u}_1 \stackrel{?}{=} 0$

$$\begin{bmatrix} -3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \frac{-3+1-1+3}{2} = 0 \checkmark$$

likewise, $\vec{x}_\perp \cdot \vec{u}_2 = 0 \checkmark$